Elastic Theorems and Energy Principles:- (1)

When an external load acts on a structure, the structure undergoes deformation and hence the work is done. To resist these external forces, the internal forces develop gradually from zero to their final value and the work is done. This internal work, which is stored as energy is due to the straining of the material and hence is called strain energy

When equillibrium is reached, as per the well known law of conservation of energy, the work done by the external forces must equal the strain energy stored.

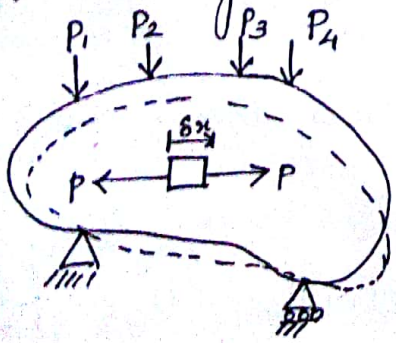
The various methods

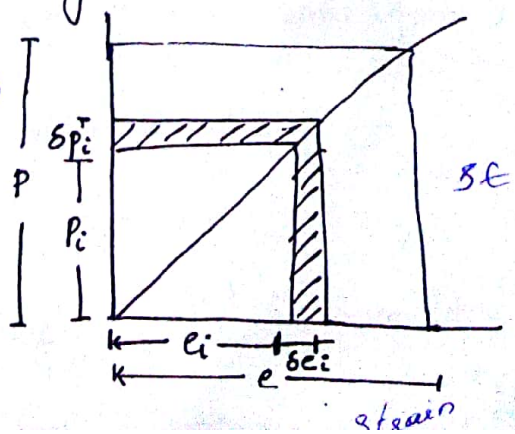
1. Strain energy / Real work method
2. Virtual work / Unit load method
3. Castigliano's method

Strain Energy and Complementary Energy:-

Energy stored in the material when it is strained or loaded within the elastic deformation

Consider the general structural system





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Let δa = cross-sectional area of the element

δx = Length "

δv = Volume "

δp_i = Stress "

δe_i = Strain "

The stress in the element gradually increases from 0 to P

The strain " " 0 to e

$\therefore$ Work done on the displacement = Force x displacement

$= P_i \delta a \cdot \delta e_i \delta x$ (area x length = V) $\quad \delta v = \delta a \, \delta x$

$= P_i \, \delta e_i \, \delta v$ (Stress x Strain x change in volume)

$\therefore$ Strain Energy stored in the element

$= \int_0^e P_i \, \delta e_i \, \delta v$ (stress x change in strain x change in volume)

= area under stress-strain curve x dv

Total work done

If the stress strain curve is leniar, the strain energy in the element $= \frac{1}{2} p e \cdot dv$

$\therefore$ Strain energy stored in the structure, U

$U = \int \frac{1}{2} p \cdot e \cdot dv$

$= \int \frac{1}{2} \times \text{stress} \times \text{strain} \times dv$

Complementary energy of the element $= \int_0^{P_i} e_i \, \delta p_i \, \delta v$

= area above stress strain curve x dv

$= \int \frac{1}{2} pe \cdot dv$ (If linear)

Complementary Energy of entire structure, $U_s = \int \frac{1}{2} p e \, dv$

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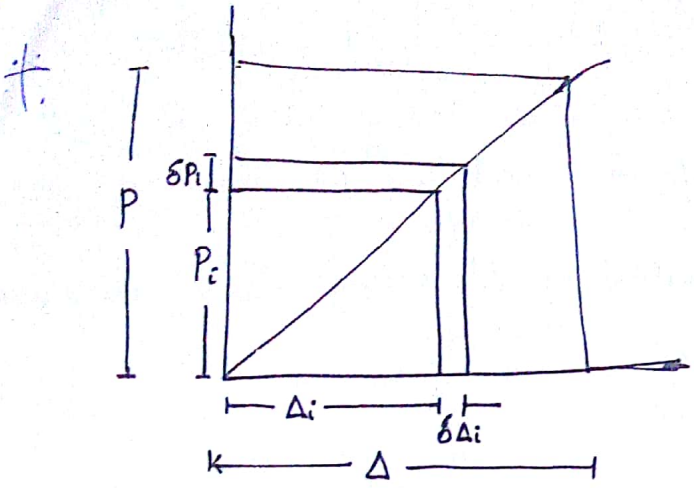
Consider Load - deformation curve



Work done by the load = $\int P_i \, \delta\Delta_i$

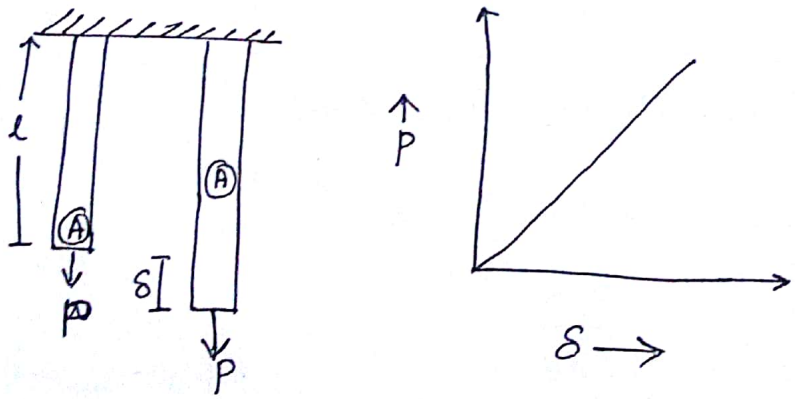
= area under load deformation curv

$$WD(V) = \frac{1}{2} \times P \cdot \Delta.$$

$\therefore$ Strain Energy = $\Sigma \frac{1}{2} P\Delta.$

Complementary Energy = $\Sigma \frac{1}{2} P\Delta.$

Strain Energy due to Axial load:



Consider a member subj of length 'l', area 'A', subjected to an external axial load 'P'. Let the extension of the member be δ.

Since the load is gradually increased from zero

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to the value 'P', the member also gradually from zero to δ

∴ The workdone by the load on the member in stretching = The product of the average load x displacement

∴ Workdone = $\frac{1}{2} P \cdot \delta$

We know that within the elastic limit

$$E = \frac{f}{\varepsilon} =$$

where $f = P/A$ (stress)

$$\varepsilon = \frac{l}{\delta}$$

$$E = \frac{P/A}{l/\delta}$$

$$E = \frac{P}{A} \times \frac{l}{\delta}$$

$$\Rightarrow \delta = \frac{Pl}{AE}$$

Workdone = Strain energy stored.

$$= \frac{1}{2} P\delta = \frac{1}{2} \times P \times \frac{Pl}{AE}$$

$$= \frac{Pl^2}{2AE}$$

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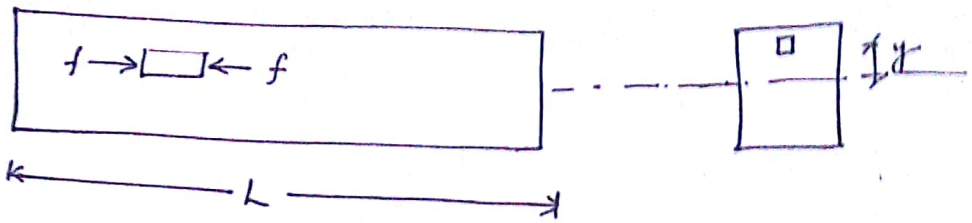
Strain Energy due to bending:-

Consider a beam subjected to pure bending. Let the area of the element be δa and it is at a distance of 'y' from NA



From the flexural theorem, bending stress, f

$$\frac{f}{y} = \frac{M}{I} = \frac{E}{R}$$

$$\Rightarrow f = \left(\frac{M}{I}\right) y$$

$$\Rightarrow \text{strain, } e = \frac{f}{E} = \left(\frac{M}{EI}\right) y$$

$\therefore$ Strain energy in the element, $= \frac{1}{2} \times \text{stress} \times \text{strain} \times dv$

$$= \frac{1}{2} \times \left(\frac{M}{I}\right) y \times \left(\frac{M}{EI}\right) y \cdot dv$$

$$= \frac{1}{2} \frac{M^2 y^2}{E I^2} dv$$

$\therefore$ Total strain energy in the beam, $= \int \frac{1}{2} \frac{M^2 y^2}{E I^2} \cdot dv$

$$= \int_0^L \int_0^A \frac{1}{2} \frac{M^2 y^2}{E I^2} \times \delta a \cdot dx$$

$$= \int_0^L \frac{M^2}{2EI^2} \left[\int_0^A y^2 \delta a \right] dx$$

$$\int_0^A y^2 \delta a = I$$

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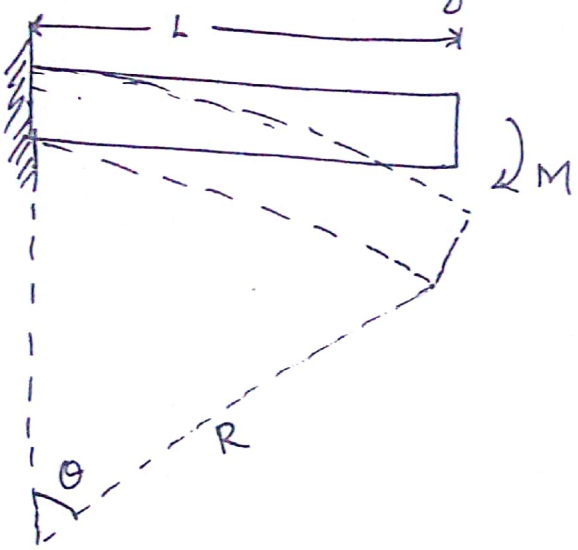
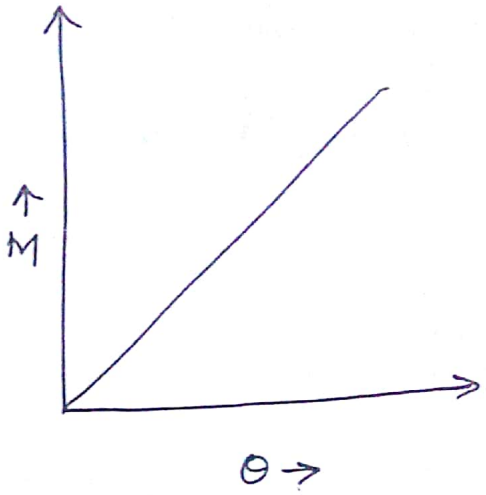
$$= \int_0^L \frac{M^2}{2EI^2} \times I \cdot dx$$

Total strain $U = \int_0^L \frac{M^2}{2EI}\, dx$

Consider a beam of Length L subjected to BM





Work done = area under the Mθ curve

$= \frac{1}{2} M\theta$

$= \frac{1}{2} \cdot M \times \frac{ML}{EI}$

Total work done $= \frac{1}{2} \frac{M^2 L}{EI}$

$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$

$R = L/\theta$

$\frac{M}{I} = \frac{E}{L/\theta}$

$\Rightarrow \theta = \frac{ML}{EI}$.

By considering a small element, total strain Energy, $U = \int_0^L \frac{M_x^2\, dx}{2EI}$

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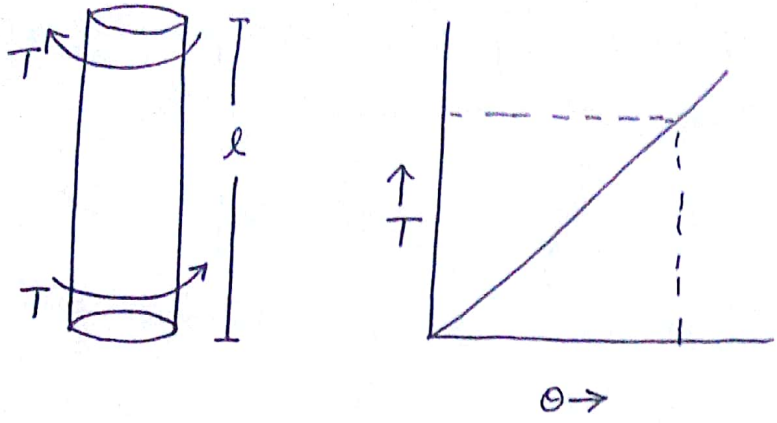
Strain Energy due to Torsion:-

Consider a shaft of l subjected to a torsional moment T



Workdone = $\frac{1}{2} T\theta$

$= \frac{1}{2} T \times \frac{Tl}{GJ}$

$\frac{T}{J} = \frac{G\theta}{l} = \frac{q}{R}$

$\theta = \frac{Tl}{GJ}$

Strain energy = $\frac{T^2 l}{2GJ}$

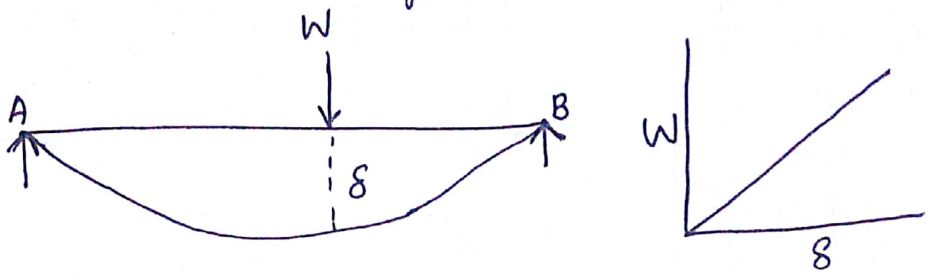
J = polar moment of inertia

Strain Energy due to Shear

Workdone by a force on a member

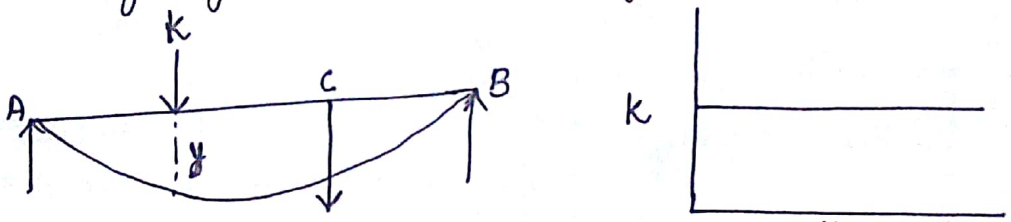
Let

i) If a load acts on a member and produces a dyle δ in its line of action by virtue of its own direct action, the work done by the load $\boxed{W = \frac{1}{2} W\delta}$



ii) If a member subjected to a load K is given a deformation y in the line of action of K by virtue of some other external agency, the workdone by the load $\boxed{K = Ky}$

Virtual work →



Consider a member carrying a load K. This load by virtue of its own direct action will produce a dyleetion Dy K is very small, dyleetion produced by k is also small and hence work done by K may be ignored.

Suppose now due to the action of some other external agency the member subjected to a deformation

Let y be the dyleetion in the line of action of load K

$\therefore$ Work done by the load $K = Ky$

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Deflection by strain Energy method :-

This method also called real work method, since work done by the actual loads are considered.

From the law of conservation of energy

Strain Energy, U = Real work done by loads.

$$U = \sum_{0}^{n} \frac{1}{2} P\Delta$$

This equation can be used to find deflection in beams and frames subject to bending stress.

$$U = \sum_{0}^{n} \frac{1}{2} P\Delta = \int \frac{M_x^2 dx}{2EI}$$

Limitations of strain Energy Method :-

Strain energy method can be conveniently used for finding deflection in structures only under the following situation

1. The structure is subjected to a single concentrated load
2. Deflection required is at the loaded point and is in the direction of the load.

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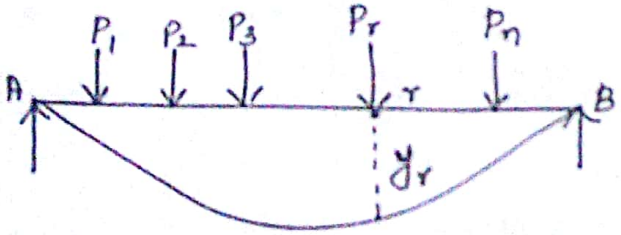
Castigliano's method for deflection :-

The first Theorem of Castigliano

In any beam or truss subjected to any load syste the deflection at any point r is given by the partial differential coefficient of the total strain energy stored with respect to a force P_r, acting at the point r in the direction in which the deflection is desired.

Consider a structure carrying a load system $P_1, P_2, \ldots P_r \ldots$



Let the deflection at the point r be y_r

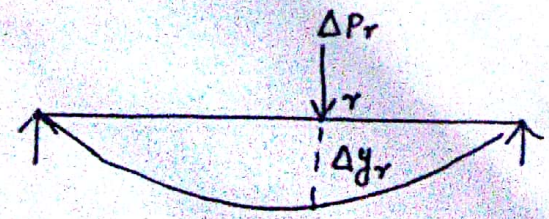
Let We = External workdone by the given load system

Wi = Strain Energy stored

$\therefore We = Wi$

Let the load ΔP_r alone be applied at r

Let the deflection at r due to the load ΔP_r be Δy_r



$\therefore$ Workdone $= \frac{1}{2} \Delta P_r \cdot \Delta y_r$

product of infinitely small quantities and can be ignore

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As the structure is carrying the load ΔP_r at r, let the given load system $P_1, P_2, P_3, \ldots P_r, \ldots P_n$ be applied on the structure

Total External work done = $W_e + \Delta P_r y_r$

Let the corresponding strain energy stored in the structure be $W_i + \Delta W_i$

$\therefore W_e + \Delta P_r \cdot y_r = W_i + \Delta W_i$ $\qquad W_e = W_i$

$\Delta P_r \, y_r = \Delta W_i$

$\Rightarrow y_r = \dfrac{\Delta W_i}{\Delta P_r}$

The above eqn is justified only when ΔP_r is an infinitely small quantity

$$y_r = \lim_{\Delta P_r \to 0} \frac{\Delta W_i}{\Delta P_r}$$

$$= \frac{\partial W_i}{\partial P_r}$$

= Partial differential coefficient of the total strain Energy stored with respect to P_r

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